handout for integration with several variables

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The integral of a function of several variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$

$$\int_{\mathcal{R}} f(\mathbf{x}) d\mathbf{x} \equiv \int_{\mathcal{R}} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \tag{1}$$

is the volume of $n+1$ the dimensional region lying 'vertically above' the domain of integration $\mathcal{R} \subset I\!R^n$ and 'below' the function $f(\mathbf{x})$.

1 Separable integrals

The most easy case is when we can separate the integration, e.g. in 2 d,

$$\int_{x=a}^{b} \int_{y=c}^{d} f(x) g(y) dx dy = \int_{x=a}^{b} f(x) dx \int_{y=c}^{d} g(y) dy \tag{2}$$

2 Iterated integration

A little more complicated are the cases, in which integration can be done by iteration, 'from inside out'. Suppose we can write the 2-d region $\mathcal{R}$ as the set $a < x < b$ and $c(x) < y < d(x)$ then we can write

$$\int_{\mathcal{R}} f(y,x) dy dx = \int_{x=a}^{b} \left[\int_{y=c(x)}^{d(x)} f(y,x) dy \right] dx \tag{3}$$

The first step is evaluate the inner integral, where we interpret $f(y,x)$ as a function of y with fixed parameter x. Suppose we can find F such that $\partial F(y,x)/\partial y = f(y,x)$, then the result of the inner integral is

$$\int_{y=c(x)}^{d(x)} f(y,x) dy = F(d(x),x) - F(c(x),x) \tag{4}$$

The result, which we call $g(x)$ is obviously a function of x only,

$$g(x) \equiv F(d(x),x) - F(c(x),x) \tag{5}$$

The next step is the outer integral, which is now just a one-dimensional integral of the function g,

$$\int_{x=a}^{b} \left[\int_{y=c(x)}^{d(x)} f(y,x) dy \right] dx = \int_{x=a}^{b} g(x) dx \tag{6}$$

Now suppose that the same 2-d region $\mathcal{R}$ can also be written as the set $s < y < t$ and $u(y) < x < v(y)$, then we can also choose to evaluate the integral as

$$\int_{\mathcal{R}} f(y,x) dx dy = \int_{y=s}^{t} \left[\int_{x=u(y)}^{v(y)} f(y,x) dx \right] dy \tag{7}$$

following the same procedure as above. In most regular cases the result is the same (for exceptions, see below (*)).

Integration with more than two variables can be done with exactly the same procedure, 'from inside out'.

2.1 Example

Integrate the function $f(x,y) = x^2y$ over the domain $\mathcal{R} = \{(x,y)|x > 0, y > 0, x+y < 1\}$.

Method 1: First integrate over y, then over x.

$\mathcal{R}$ can be described as $\{(x,y)|0 < x < 1, 0 < y < 1-x\}$

$$\begin{aligned}
\int_{\mathcal{R}} f(x,y)dxdy &= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} x^2 y dy\right] dx && (8)\\
&= \frac{1}{2}\int_{x=0}^{1} x^2(1-x)^2 dx && (9)\\
&= \frac{1}{2}\int_{x=0}^{1} x^4 - 2x^3 + x^2 dx && (10)\\
&= \frac{1}{2}[\frac{1}{5} - \frac{2}{4} + \frac{1}{3}] && (11)\\
&= 1/60 && (12)
\end{aligned}$$

Method 2: First integrate over x, then over y.

$\mathcal{R}$ can be described as $\{(x,y)|0 < y < 1, 0 < x < 1-y\}$

$$\begin{aligned}
\int_{\mathcal{R}} f(x,y)dxdy &= \int_{y=0}^{1}\left[\int_{x=0}^{1-y} x^2 y dx\right] dy && (13)\\
&= \frac{1}{3}\int_{y=0}^{1} (1-y)^3 y dy && (14)\\
&= \frac{1}{3}\int_{y=0}^{1} y - 3y^2 + 3y^3 - y^4 dy && (15)\\
&= \frac{1}{3}[\frac{1}{2} - 3\frac{1}{3} + 3\frac{1}{4} - \frac{1}{5}] && (16)\\
&= 1/60 && (17)
\end{aligned}$$

2.2 Exceptions (*)

There exists functions for which different orders of integration give different results. In those cases the double integral is actually ill defined. An example is the integral

$$\int_{x=0}^{1}\left(\int_{y=0}^{1}\frac{x-y}{(x+y)^3}dy\right)dx = \int_{x=0}^{1}\frac{dx}{(1+x)^2} = \frac{1}{2} \tag{18}$$

If the order of integration is reversed, the sign is flipped and the result is $-\frac{1}{2}$. There is a theorem that states that if the iteration of integration can be done for the absolute value of the function $|f(y,x)|$, then the double integral is well defined, and the ordering of variables does not matter. So if f is positive, one may always reverse the order of integration.

3 Transformation of variables

Often it is easier to do the multidimensional integral in another coordinate frame. This is an generalization of the rule for change of variables in one dimension.

3.1 One variable

Suppose we want to do the integration

$$\int_{y=c}^{d} f(y)dy \tag{19}$$

but the function $f(y)$ is easier expressed as a $f(y(x))$ which is a function of x. So we want to use x as integration variable. If where y and x are related via invertible differentiable mappings $y = y(x)$ and $x = x(y)$ and the end points of the interval $(y = c, y = d)$ are mapped to $(x = a, x = b)$, (so $c = y(a)$, etc) then we have the equality

$$\begin{aligned} \int_{y=c}^{d} f(y)dy &= \int_{y(x)=c}^{d} f(y(x))dy(x) && (20)\\ &= \int_{x=a}^{b} f(y(x))y'(x)dx && (21) \end{aligned}$$

The derivative $y'(x)$ comes in as the ratio between the lengths of the differentials dy and dx,

$$dy = y'(x)dx \tag{22}$$

3.2 Several variables

With several variables, the substitution rule is generalized as follows. We have the invertible mapping $\mathbf{y} = \mathbf{y}(\mathbf{x})$. Let us also assume that the region of integration of $\mathcal{R}$ is mapped by to $\mathcal{S}$, (so $\mathcal{S} = \mathbf{y}(\mathcal{R})$), then we have the equality

$$\begin{aligned} \int_{\mathbf{y}\in\mathcal{S}} f(\mathbf{y})d\mathbf{y} &= \int_{\mathbf{y}(\mathbf{x})\in\mathcal{S}} f(\mathbf{y})d\mathbf{y}(\mathbf{x}) && (23)\\ &= \int_{\mathbf{x}\in\mathcal{R}} f(\mathbf{y}(x)) \left|\det\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right)\right| d\mathbf{x} && (24) \end{aligned}$$

The factor $\det\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right)$ is called the Jacobian of the coordinate transformation. Written out in more detail

$$\det\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right) = \begin{vmatrix} \frac{\partial y_1(\mathbf{x})}{\partial x_1} & \frac{\partial y_1(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial y_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial y_2(\mathbf{x})}{\partial x_1} & \frac{\partial y_2(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial \mathbf{y}_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y_n(\mathbf{x})}{\partial x_1} & \frac{\partial y_n(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial y_n(\mathbf{x})}{\partial x_n} \end{vmatrix} \tag{25}$$

The *absolute value* [1] of the Jacobian comes in as the ratio between that the volume represented by the differential $d\mathbf{y}$ and the volume represented by the differential $d\mathbf{x}$, i.e.,

$$d\mathbf{y} = \left|\det\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right)\right| d\mathbf{x} \tag{26}$$

As a last remark, it is good to know that

$$\det\left(\frac{\partial \mathbf{x}(\mathbf{y})}{\partial \mathbf{y}}\right) = \det\left(\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1}\right) = \frac{1}{\det\left(\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}\right)} \tag{27}$$

[1] In the single-variable case, we took the orientation of the integration interval into account ($\int_a^b f(x)dx = -\int_b^a f(x)dx$). With several variables, this is awkward. Fortunately, it turns out that the orientation of the mapping of the domain always cancels to the 'orientation' of the Jacobian (= sign of the determinant). Therefore we take a positive orientation and the absolute value of the Jacobian

3.3 Important example: polar coordinates

3.3.1 Area of a disc

Consider a two-dimensional disc with radius R

$$D = \{(x,y)|x^2 + y^2 < R^2\} \tag{28}$$

Its area is

$$\int_D dxdy \tag{29}$$

This integral is easiest evaluated by going to 'polar-coordinates'. The mapping from polar coordinates (r,θ) to Cartesian coordinates (x,y) is

$$x = r\cos\theta \tag{30}$$

$$y = r\sin\theta \tag{31}$$

Since In polar coordinates, the disc is described by $0 \leq r < R$ (since $x^2 + y^2 = r^2$) and $0 \leq \theta < 2\pi$.

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \tag{32}$$

In other words,

$$dxdy = rdrd\theta \tag{33}$$

The area of the disc is now easily evaluated.

$$\int_D dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} rdrd\theta = \pi R^2 \tag{34}$$

3.3.2 Gaussian integral in 2 d

The integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-(x^2+y^2))dxdy \tag{35}$$

is easily evaluated by going to 'polar-coordinates',

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-(x^2+y^2))dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \exp(-r^2)rdrd\theta \tag{36}$$

$$= 2\pi\int_{r=0}^{\infty} \exp(-r^2)rdr \tag{37}$$

$$= \pi \tag{38}$$