Binary neurons with sequential dynamics

1. (a) The network has four states $s = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. For each state, we can compute the local field $h_i = \sum_j w_{ij} s_j$ or $h_1 = ws_2$ and $h_2 = ws_1$. The probality to make a transition from state s to state s' is only non-zero when s and s' differ by at most one bit. When they differ one bit at neuron i, the probability is given by

$$T(s'|s) = \frac{1}{2}\sigma(s'_i h_i(s)) = \frac{1}{2}\sigma(-ws_1s_2)$$

The factor $\frac{1}{2}$ is due to the random choice of the selection of the neuron that is tossed. The probability to stay in the same state is

$$T(s|s) = 1 - 2\frac{1}{2}\sigma(-ws_1s_2) = \sigma(ws_1s_2)$$

Thus, T is given by

$$T(s'|s) = \begin{pmatrix} \sigma(w) & \frac{1}{2}\sigma(w) & \frac{1}{2}\sigma(w) & 0 \\ \frac{1}{2}\sigma(-w) & \sigma(-w) & 0 & \frac{1}{2}\sigma(-w) \\ \frac{1}{2}\sigma(-w) & 0 & \sigma(-w) & \frac{1}{2}\sigma(-w) \\ 0 & \frac{1}{2}\sigma(w) & \frac{1}{2}\sigma(w) & \sigma(w) \end{pmatrix}$$

(b) For $w \to \infty$, T becomes

$$T(s'|s) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

Setting up the eigenvalue equation with eigenvalue 1: $T\vec{r} = \vec{r}$, with $\vec{r}$ the stationary distribution over the 4 states s gives $r_2 = r_3 = 0$. Thus, the eigenspace for eigenvalue 1 is two dimensional. This means that the network is non-ergodic in the limit $w \to \infty$. We can choose any basis that spans this two dimensional space. A particularly convenient choice is to use as the first basis vector $\vec{r_1}$ the Boltzmann distribution Eq. **??** and choose the second vector orthogonal to it:

$$\vec{r_1} = \frac{1}{2}(1,0,0,1) \quad \vec{r_2} = \frac{1}{2}(1,0,0,-1)$$

(c) The invariants are given by the left eigenvectors with eigenvalue 1, which we denote by $\vec{l}$. Then $\vec{l}T = \vec{l}$ gives $\frac{1}{2}(l_1 + l_4) = l_2 = l_3$. We can find two vectors $\vec{l_1}$ and $\vec{l_2}$ that are in this subspace and simulaneously satisfy the duality relation eq. **??**:

$$\vec{l_1} = (1,1,1,1) \quad \vec{l_2} = (1,0,0,-1)$$