Binary neurons with sequential dynamics

1. (a) The network has four states \( s = \{(1,1), (1,-1), (-1,1), (-1,-1)\} \). For each state, we can compute the local field \( h_i = \sum_j w_{ij}s_j \) or \( h_1 = ws_2 \) and \( h_2 = ws_1 \). The probability to make a transition from state \( s \) to state \( s' \) is only non-zero when \( s \) and \( s' \) differ by at most one bit. When they differ one bit at neuron \( i \), the probability is given by
\[
T(s'|s) = \frac{1}{2} \sigma(s'_i h_i(s)) = \frac{1}{2} \sigma(-ws_1s_2)
\]
The factor \( \frac{1}{2} \) is due to the random choice of the selection of the neuron that is tossed. The probability to stay in the same state is
\[
T(s|s) = 1 - \frac{1}{2} \sigma(-ws_1s_2) = \sigma(ws_1s_2)
\]
Thus, \( T \) is given by
\[
T(s'|s) = \begin{pmatrix}
\sigma(w) & \frac{1}{2} \sigma(w) & \frac{1}{2} \sigma(w) & 0 \\
\frac{1}{2} \sigma(-w) & \sigma(-w) & 0 & \frac{1}{2} \sigma(-w) \\
\frac{1}{2} \sigma(-w) & 0 & \sigma(-w) & \frac{1}{2} \sigma(-w) \\
0 & \frac{1}{2} \sigma(w) & \frac{1}{2} \sigma(w) & \sigma(w)
\end{pmatrix}
\]
(b) For \( w \to \infty \), \( T \) becomes
\[
T(s'|s) = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 1
\end{pmatrix}
\]
Setting up the eigenvalue equation with eigenvalue 1: \( T\vec{r} = \vec{r} \), with \( \vec{r} \) the stationary distribution over the 4 states \( s \) gives \( r_2 = r_3 = 0 \). Thus, the eigenspace for eigenvalue 1 is two dimensional. This means that the network is non-ergodic in the limit \( w \to \infty \). We can choose any basis that spans this two dimensional space. A particularly convenient choice is to use as the first basis vector \( \vec{r}_1 \) the Boltzmann distribution Eq. ?? and choose the second vector orthogonal to it:
\[
\vec{r}_1 = \frac{1}{2}(1,0,0,1) \quad \vec{r}_2 = \frac{1}{2}(1,0,0,-1)
\]
(c) The invariants are given by the left eigenvectors with eigenvalue 1, which we denote by \( \vec{l} \). Then \( \vec{l} T = \vec{l} \) gives \( \frac{1}{2}(l_1 + l_4) = l_2 = l_3 \). We can find two vectors \( \vec{l}_1 \) and \( \vec{l}_2 \) that are in this subspace and simultaneously satisfy the duality relation eq. ??:
\[
\vec{l}_1 = (1,1,1,1) \quad \vec{l}_2 = (1,0,0,-1)
\]