The storage capacity of the Hopfield network

In this exercise you will estimate numerically the storage capacity of the Hopfield network at zero temperature ($\beta = \infty$). Consider a neural network of size $N$. Choose $N$ fixed as large as possible (I used $N = 1000$, but you may want to start with smaller $N$). Choose $P$ random patterns $\xi_{\mu} = \pm 1, i = 1, \ldots, N, \mu = 1, \ldots, P$. Vary $P$ from $0.02N$ to $0.4N$ in steps of $0.02N$. Construct the weight matrix $w_{ij} = \frac{1}{N} \sum \xi_{\mu} \xi_{\nu} = \delta_{ij}$. Set the diagonal elements $w_{ii} = 0$. For each $P$ run $N_r$ runs (I used $N_r = 50$) using sequential zero temperature Glauber dynamics.

For each $P$ and each run $r$ do the following:
1: Initialize $s_i = \xi_{\mu=1}^i$ and flip 10% of the bits of $s_i \rightarrow -s_i$
2: Run sequential Glauber dynamics $s'_i = \text{sign} \left( \sum_j w_{ij} s_j \right)$ until convergence.
3: Compute the overlap between the converged state $s$ and the memory $\xi_{\mu=1}^i$: $M(P, r) = \frac{1}{N} \sum_{i=1}^N \xi_{\mu=1}^i s_i$

Plot the average overlap $M(\alpha) = \frac{1}{N_r} \sum_r M(P, r)$ versus $\alpha = P/N$. The result should look like Fig. 1. For $N = 200, 1000$ the total time to produce these plots in Matlab was 2 sec and 156 sec, respectively on a 3 year old Macbook. The overlap $M(\alpha)$ drops sharply for $\alpha \approx \alpha_c(N)$. The theoretical prediction is that $\lim_{N \to \infty} \alpha_c(N) = 0.138$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Memory of the Hopfield network as a function of $\alpha = P/N$, for $N = 200$ and $N = 1000$. Results are average over $N_r = 50$ runs.}
\end{figure}

\footnote{It is important to use sequential Glauber dynamics and not parallel dynamics and to use $w_{ii} = 0$, because only in that case is the equilibrium given by the Boltzmann distribution $p(s) = \frac{1}{Z} e^{\beta H(s)}$, where $Z = \sum_s e^{\beta H(s)}$.}